Lec 18: 03/26/2012

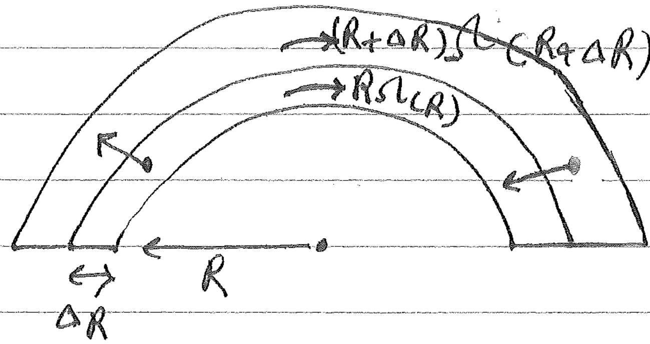
Accretion Disk Theory:

Whether a compact object accrets from the surrounding medium, or from a binary companion, the accreting plasma settles into a disk perpendicular to its net angular momentum vector. Even if the accretion proceeds spherically at first, cooling processes eventually dissipate the plasma's support in the parallel direction away.

Thin-Disk Theory:

To describe the structure of the disk, we first find hydrodynamic equations similar to those derived in the case of spherical accretion. We will make full use of the specific geometry in the Keplerian motion.

Consider a thin disk rotating close to an orbital plane



At any radius R, the matter rotates in a ring with a circular velocity $v_\phi(R) = R\Omega_K(R)$, where:

$$\Omega_K(R) \equiv \left(\frac{GM}{R^3}\right)^{1/2}$$

This is the Keplerian angular velocity. The mass and angular momentum contents of the ring at radius R are:

mass $= 2\pi R \Delta R \Sigma$

angular momentum $= (2\pi R \Delta R \Sigma) R^2 \Omega_K(R)$

Here $\Sigma \equiv \rho H$ is the surface density; with H being the thickness of the disk.

The equation for the conservation of mass is:

$$R\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial R}(R\Sigma v_R) = 0$$

v_R is the radial velocity (inward) due to accretion.

The equation for the angular momentum is:

$$R\frac{\partial}{\partial t}(\Sigma R^2\Omega)+\frac{\partial}{\partial R}(R\Sigma v_R R^2\Omega)=-\frac{1}{2\pi}\frac{\partial T_{out}}{\partial R}$$

Here T_{out} is the viscous torque that a ring exerts to the neighboring outer ring, which is given by:

$$T_{out}\sim 2\pi R\Sigma\tilde{v}\,R(R+\lambda)[\Omega(R)-\Omega(R+\lambda)]$$

λ and $\tilde{v}$ are the length scale and speed, respectively, for matter crossing between neighboring rings (due to thermal motion or turbulence). By defining the coefficient of kinematic viscosity $v\equiv\lambda\tilde{v}$, we have:

$$T_{out}\sim -2\pi v\Sigma R^3\Omega'$$

Using this expression and the equation from the mass conservation, the angular momentum equation can be written as follows:

$$R\Sigma v_R(R^2\Omega)'=\frac{\partial}{\partial R}(Rv\Sigma R^3\Omega')$$

For a Keplerian flow $\Omega = \Omega_k = \left(\frac{GM}{R^3}\right)^{\frac{1}{2}}$, which results in,

$$v_R = \frac{-3}{\Sigma R^{1/2}} \frac{\partial}{\partial R}\left(\nu \Sigma R^{\frac{1}{2}}\right)$$

In situations where ν is Constant and the disk is in steady state, the mass Conservation implies that $R\Sigma v_R = \text{Const.}$, and hence the mass accretion rate is:

$$\dot{M} = 2\pi R \Sigma (-v_R) = \text{Const.}$$

Also:

$$v_R \sim O\left(\frac{\nu}{R}\right) \Rightarrow \frac{v_R}{c_s} \sim \alpha \left(\frac{H}{R}\right)$$

Here we have used the useful parameterization $\nu \equiv \alpha c_s H$.

For thin disks $H \ll R$, and hence $v_R \ll c_s$, which validate the assumption of quasi-Keplerian motion of the disk.

The power exerted on the ring at radius R due to the net viscous torque is,

$$P = -\Omega \frac{d\,\Gamma_{out}}{dR}\, dR$$

It can be written as:

$$P = -\left[\frac{d}{dR}\left(\Gamma_{out}\,\Omega\right) - \Gamma_{out}\,\Omega'\right] dR$$

The first term inside the brackets simply gives a (global) transfer of rotational energy through the disk. The second term represents an actual local dissipation that produces heat. As long as the dissipated energy (coming from the conversion of gravitational potential energy) is radiated efficiently, the disk will remain thin. Otherwise, the thin-disk approximation is no longer valid. We will discuss this in more detail later on.

Assuming that the dissipated energy is radiated from the surface of the disk (i.e., disk is optically thick), the dissipation rate per unit surface area is;

$$D(R) = \frac{T_{out}\,\Omega'}{4\pi R} = -\frac{1}{2}\nu\Sigma(R\Omega')^2$$

Using the mass conservation and angular momentum equations we find:

$$R\Sigma v_R\, R^2\Omega = -\frac{1}{2\pi}T_{out} + \text{Const.} \Rightarrow \nu\Sigma\Omega' = \Sigma(-v_R)\Omega + \frac{\text{Const.}}{R^3}$$

Near the surface of the compact object, Ω eventually merges with its rotation, implying that $\Omega' \to 0$ and $\Omega = \Omega_K(R_*)$ (R_* being the object's radius). Thus:

$$-\frac{\dot{M}}{2\pi}(GMR_*)^{1/2} = \text{Const.}$$

Finally, we arrive at an expression for ν as a function of $\dot{M}$:

$$\nu\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]$$

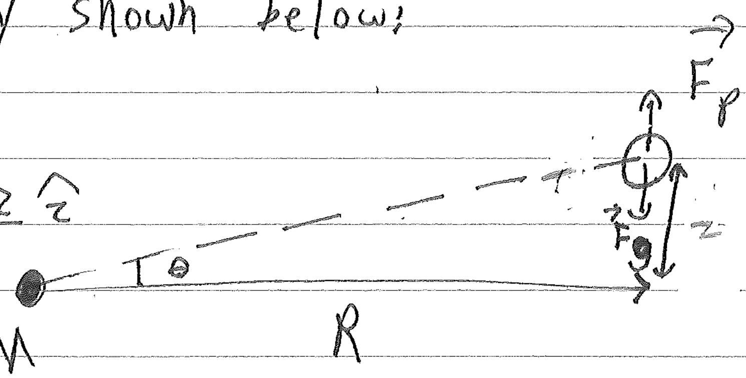
This results in:

$$D(R) = \frac{3GM\dot{M}}{8\pi R^3}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]$$

This expression for the dissipation rate as a function of physical parameters is useful when calculating the spectrum of radiation from a thin disk.

To verify the thinness of the disk, let us consider the vertical structure of the disk. We may assume that the gas is in local hydrostatic equilibrium in the vertical direction as schematically shown below:

$$\vec{F}_g \approx -\frac{GM\rho}{R^2}\sin\theta\,\hat{z} = -\frac{GM\rho z}{R^3}\,\hat{z}$$



$$\vec{F}_p = -\frac{\partial P}{\partial z}\,\hat{z}$$

$$\vec{F}_g + \vec{F}_p = 0 \Rightarrow \frac{1}{\rho}\frac{dP}{dz} = -\frac{GMz}{R^3}$$

If the gas is isothermal in the vertical direction (or the temperature gradient is sufficiently small, as is usually the case), we have:

$$\frac{dP}{dz} = \frac{R_g}{\nu} T(R) \frac{d\rho}{dz} = \frac{P}{\rho}\frac{d\rho}{dz} = -\frac{GMz\rho}{R^3}$$

Here we have used the equation of state of the gas, R_g is the gas constant, and ν is the mean molecular weight per particle ($\nu \approx 1$ for the Hydrogen atom).

The above equation can be easily solved, resulting in:

$$\rho(R,z) = \rho_c(R) \exp\left(\frac{-z^2}{2\zeta^2}\right)$$

Here:

$$\zeta \equiv \left(\frac{R^3 R_g T}{GM\nu}\right)^{\frac{1}{2}}$$

It can be interpretted as $\frac{H}{2}$, which is half of the disk thickness. To put things in perspective, consider:

$$\frac{\zeta}{R} \sim 10^{-7} R^{\frac{1}{2}} \qquad (T \sim 10^4 - 10^5 K)$$

For $R \sim 10$ km (as for a neutron star), we then find $\zeta \sim 10^{-4} R$. For $R \sim 10^4$ km (as for a white dwarf), we have $\zeta \sim 3\times10^{-3} R$. The thin-disk geometry is justified in both cases.